Field and Galois Theory
(Mid-Sem Test, BMath-3rd year, 2025)

Instructions: Total time 3 Hour. Solve problems for a max score of 20. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in a homework or an exercise from the class, please supply its full solution.

1. An algebraic number is called an *algebraic integer* if it satisfies a monic polynomial in $\mathbb{Z}[X]$. Prove that **(i)** for any algebraic number α, there is an integer $n \geq 1$ such that $n\alpha$ is an algebraic integer. **(ii)** If a rational number is an algebraic integer, it must belong to $\mathbb{Z}$. (3+2)

2. Consider the field extension $\mathbb{Q}(S)/\mathbb{Q}$, where $S = \{\sqrt{p} |\ p \text{ is a prime in } \mathbb{Z}\}$. Let q be a prime in $\mathbb{Z}$. Prove that if n is any odd positive integer, $n \geq 3$, then $q^{\frac{1}{n}} \notin \mathbb{Q}(S)$. (5)

3. Let L/k be an algebraic field extension and $\phi : L \to L$ be a k-linear field homomorphism. Prove that ϕ is surjective. Does this hold if L/k is NOT algebraic? Explain. (2+3)

4. Let k be a field of positive characteristic p and $f(X) \in k[X]$ irreducible. **(i)** Show that there exists an integer $e \geq 0$ such that $f(X) = g(X^{p^e})$ for some separable irreducible polynomial $g(X) \in k[X]$. **(ii)** Deduce that every root of $f(X)$ has multiplicity p^e in a splitting field of $f(X)$. (2+3)

5. Let $\Phi_k(X)$ denote the kth cyclotomic polynomial, i.e., the minimal polynomial of a primitive kth root of unity in $\mathbb{C}$ over $\mathbb{Q}$. **(i)** Let p be a prime not dividing n. Prove that $\Phi_{pn}(X) = \frac{\Phi_n(X^p)}{\Phi_n(X)}$. **(ii)** Let n be odd. Prove that $\Phi_{2n}(X) = \Phi_n(-X)$. (3+2)

6. Let p be a prime and $K := \mathbb{F}_p(t)$ be the rational function field over $\mathbb{F}_p$. Let $\sigma : K \to K$ be the $\mathbb{F}_p$-automorphism defined by $\sigma(t) = t + 1$. Find the fixed field of the cyclic subgroup $< \sigma >$ of $\text{Aut}(K)$ generated by σ.(5)